

l^p -DECOUPLING FOR (k, p) -BROADNESS

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ABSTRACT. We prove an l^p -decoupling result for restricted (k, p) -broadness, that has an application in Fourier restriction problem.

1. INTRODUCTION

L stands for a large constant, ε denotes a tiny positive number, R represents a number larger than L^L . Let A_1 stand for R^{ε^L} and K represent a dyadic number, usually much larger than A_1 . For any dyadic number D , \mathcal{Q}_D is the collection of all dyadic cubes in \mathbb{R}^{n+1} , of side length D . We use ϕ to denote a standard bump function on a 1-neighborhood away from the origin. Let Σ stand for local parabolic surface $\{(\xi, |\xi|^2) : \xi \in \text{supp}(\phi)\}$ in \mathbb{R}^{n+1} . Σ_D denotes $\frac{1}{D^2}$ -neighborhood of Σ . We partition Σ into a union of $\frac{1}{D}$ -caps τ 's, hence divide Σ_D naturally into a collection of D^{-2} -neighborhood of caps τ 's. This collection is denoted by Θ_D . We use \mathcal{T}_D to denote the collection of those D^{-1} -caps τ 's that form a partition of Σ , and $e(\tau)$ to represent the unit normal vector at the center of the cap τ . Each $\theta \in \Theta_D$ can be viewed essentially a $\frac{1}{D} \times \cdots \times \frac{1}{D} \times \frac{1}{D^2}$ rectangular box in \mathbb{R}^{n+1} , and is generated by a cap, say τ_θ . For brevity, let $e(\theta)$ stand for $e(\tau_\theta)$. For any function $F : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ with Fourier support in Σ_D , F_θ represents Fourier restriction of F in θ , that is, $\widehat{F_\theta} = \mathbf{1}_\theta \widehat{F}$. From the partition of unity, we write

$$(1.1) \quad F(\mathbf{x}) = \sum_{\theta \in \Theta_D} F_\theta(\mathbf{x}).$$

For any $\tau \in \mathcal{T}_D$, define K_τ by

$$(1.2) \quad \widehat{K_\tau} = \Phi_\theta,$$

where θ is the D^{-2} neighborhood of τ , and Φ_θ is a standard bump function on 2θ .

For any positive integer $d \leq n+1$, let \mathcal{V}_d denote the collection of all d -dimensional subspaces in \mathbb{R}^{n+1} . For any $V_1, \dots, V_A \in \mathcal{V}_d$, let $\mathcal{T}_D(V_1, \dots, V_A)$ denote the following collection of caps:

$$(1.3) \quad \mathcal{T}_D(V_1, \dots, V_A) := \{\tau \in \mathcal{T}_D : \text{dist}(e(\tau), V_a) \geq 100D^{-1} \text{ for all } a \in \{1, \dots, A\}\}.$$

For any dyadic number D , D_0 stands for the largest dyadic number not exceeding $D^{\varepsilon^{\sqrt{L}/2}}$. For any cube $B \in \mathcal{Q}_{D^2}$, $p \geq 2$, $k \in \{2, \dots, n+1\}$ and any positive integer $A \leq A_1$, we define the restricted (k, p) -broadness of F on a subset U of B with the parameter A by

$$(1.4) \quad \|F\|_{\mathbf{Br}_{k,p}(A,U,B)} = \left(\sum_{\mathbf{q} \in \mathcal{Q}_{D_0^2} : \mathbf{q} \subset U} \min_{V_1, \dots, V_A \in \mathcal{V}_{k-1}} \sup_{\tau \in \mathcal{T}_{D_0}(V_1, \dots, V_A)} \int_{\mathbf{q}} |K_\tau * (\sum_{\theta \in \Theta_{D,V}} F_\theta)(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}},$$

where

$$(1.5) \quad \Theta_{D,V} = \{\theta \in \Theta_D : \text{dist}(e(\theta), V) \leq 100D^{-1}\}$$

for some k -dimensional space $V \in \mathcal{V}_k$. V here may depend on F and $B \in \mathcal{Q}_{D^2}$, but independent of τ , \mathbf{q} , A , and the variable \mathbf{x} . It is straightforward to get

$$(1.6) \quad \left\| \sum_{j=1}^N F_j \right\|_{\mathbf{Br}_{k,p}(A,U,B)} \leq N^C \sum_{j=1}^N \|F_j\|_{\mathbf{Br}_{k,p}(A/N,U,B)},$$

where C is an absolute constant. This can be proved by noticing

$$(1.7) \quad \min_{V_1, \dots, V_A \in \mathcal{V}_{k-1}} \sup_{\tau \in \mathcal{T}_D(V_1, \dots, V_A)} (F_{1,\tau} + F_{2,\tau}) \leq \min_{V_1, \dots, V_{A'} \in \mathcal{V}_{k-1}} \left(\sup_{\tau \in \mathcal{T}_D(V_1, \dots, V_{A'})} F_{1,\tau} + \sup_{\tau \in \mathcal{T}_D(V_{A'+1}, \dots, V_A)} F_{2,\tau} \right),$$

for any positive integer A' with $A' \leq A$, and nonnegative functions $F_{1,\tau}$, $F_{2,\tau}$. (1.6) serves as Minkowski inequality and henceforth the broadness behaves very similar to a norm. L can be chosen as a number going beyond $4n^2$. The tiny ε can be selected such that $\varepsilon^{L/10} C \leq 1$ for the constant C in (1.6).

We define l^p -decoupling norm of F to be

$$(1.8) \quad \|F\|_{p,D,U} = \left(\sum_{\theta \in \Theta_D} \|F_\theta\|_{L^p(U)}^p \right)^{1/p},$$

for any measurable set U in \mathbb{R}^{n+1} . Throughout the paper, for $U \subset B \in \mathcal{Q}_{D^2}$, we use U^* to denote a D_0 -dilation of U , and $\|F\|_{\mathbf{Br}_{k,p}(A,B)}$ represents $\|F\|_{\mathbf{Br}_{k,p}(A,B^*)}$.

Theorem 1.1. *Let D be a sufficiently large dyadic number and $2 \leq k \leq n$. Then for any $B \in \mathcal{Q}_{D^2}$ and $\frac{2k}{k-1} \geq p \geq \frac{2(n+1-\frac{k-1}{2})}{n-\frac{k-1}{2}}$,*

$$(1.9) \quad \|F\|_{\mathbf{Br}_{k,p}(A,B)} \leq CD^{n-\frac{2(n+1)}{p}+\varepsilon} \|F\|_{p,D,B^*}.$$

Here C is an absolute constant independent of B , A , D and F .

For any dyadic number D , and any Schwartz function F , we define the (k,p) -broadness of F on a subset U with the parameters A and D by

$$(1.10) \quad \|F\|_{\mathfrak{B}_{k,p,A,D}(U)} = \left(\sum_{\mathbf{q} \in \mathcal{Q}_{D^2}: \mathbf{q} \subset U} \min_{V_1, \dots, V_A \in \mathcal{V}_{k-1}} \sup_{\tau \in \mathcal{T}_D(V_1, \dots, V_A)} \int_{\mathbf{q}} |K_\tau * F(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}},$$

Here k is an integer between 2 and $n+1$. The (k,p) -broadness was first introduced by Guth in [5], in which L^p -estimate for the broadness was established via Guth's polynomial partitioning method, developed in [4]. The l^2 -decoupling conjecture was solved by Bourgain and Demeter [1]. Guth's estimates for the broadness, combining with Bourgain and Demeter's l^2 -decoupling result, yields new estimates [5] for L^p -boundedness of the Fourier restriction operator, which improves Bourgain-Guth's Theorem [3] on the restriction conjecture. Guth's L^2 -estimates on the broadness plus Theorem 1.1 can also imply Guth's very recent result on the restriction conjecture (see Section 6). Any improvement of the lower bound of p in Theorem 1.1 results in a new estimate for the restriction conjecture. This is our main motivation in this paper. It is natural to ask

Conjecture 1.1. *Is the following estimate true?*

$$(1.11) \quad \|F\|_{\mathbf{Br}_{k,p}(A,B)} \lesssim D^{n-\frac{2(n+1)}{p}+\varepsilon} \|F\|_{p,D,B^*}$$

for $p > \frac{2(n+1)}{n}$, $2 \leq k \leq n+1$ and any $B \in \mathcal{Q}_{D^2}$.

2. PASSING LINEAR OPERATOR TO (k, p) -BROADNESS

In this section, we aim to use the restricted broadness to control L^p -norm of a function F whose Fourier transform is supported in Σ_R . The argument used here is similar to Bourgain-Guth multilinear method in [3].

Lemma 2.1. *Let D be any dyadic number, U be a dyadic cube in \mathbb{R}^{n+1} with side length larger than D^2 , and M be an integer such that $2M^2 \leq A$. Then for $k \geq 2$ and any Schwartz function whose Fourier support is in Σ_D ,*

$$(2.1) \quad \|F\|_{\mathfrak{B}_{k,p,A,D_0}(U)} \leq D^{2n} \|F\|_{\mathfrak{B}_{k+1,p,\frac{A}{M},D}(U)} + \left(\frac{A}{M}\right)^C \Xi_{k,p}(M/2, D, U).$$

where $\Xi_{k,p}(M, D, U)$ is defined by

$$(2.2) \quad \Xi_{k,p}(M, D, U) := \left(\sum_{B \in \mathcal{Q}_{D^2}: B \subset U} \|F\|_{\mathbf{B}_{k,p}(M,B,B)}^p \right)^{\frac{1}{p}}.$$

Proof. For any given cube $B \in \mathcal{Q}_{D^2}$, take $V'_1, \dots, V'_{A/M} \in \mathcal{V}_k$ to be the minimizer obeying

$$(2.3) \quad \sup_{\tau_1 \in \mathcal{T}_D(V'_1, \dots, V'_{A/M})} \int_B |K_{\tau_1} * F|^p = \min_{V_1, \dots, V_{A/M} \in \mathcal{V}_k} \sup_{\tau_1 \in \mathcal{T}_D(V_1, \dots, V_{A/M})} \int_B |K_{\tau_1} * F|^p.$$

Because of (1.6), the (k, p) -broadness $\|F\|_{\mathfrak{B}_{k,p,A,D}(U)}$ can be dominated by

$$(2.4) \quad \left(\sum_{\substack{B \in \mathcal{Q}_{D^2} \\ B \subset U}} \left\| \sum_{\tau_1 \in \mathcal{T}_D(V'_1, \dots, V'_{A/M})} K_{\tau_1} * F \right\|_{\mathfrak{B}_{k,p,\frac{A}{2},D_0}(B)}^p \right)^{\frac{1}{p}} + \left(\sum_{B \subset U} \left\| \sum_{\tau_1 \notin \mathcal{T}_D(V'_1, \dots, V'_{A/M})} K_{\tau_1} * F \right\|_{\mathfrak{B}_{k,p,\frac{A}{2},D_0}(B)}^p \right)^{\frac{1}{p}}.$$

The first term is clearly bounded by

$$(2.5) \quad D^{2n} \|F\|_{\mathfrak{B}_{k+1,p,A/M,D}(U)}.$$

Notice that $\tau_1 \notin \mathcal{T}_D(V'_1, \dots, V'_{A/M})$ implies that $e(\tau_1)$ lies in a $1/D$ -neighborhood of V'_a for some $a \in \{1, \dots, A/M\}$. Using (1.6), we control the second term in (2.4) by

$$(2.6) \quad \left(\frac{A}{M}\right)^C \Xi_{k,p}(M, D, U).$$

Therefore the proof is completed. \square

Lemma 2.2. *Let F be a Schwartz function with Fourier support in Σ_R , D be any dyadic number such that $R \geq D \geq R^{\varepsilon_L}$, and $U \in \mathcal{Q}_{R^2}$.*

$$(2.7) \quad \|F\|_{L^p(U)} \lesssim D^{2n} \|F\|_{\mathfrak{B}_{2,p,A,D}(U)} + A \left\| \sup_{\tau \in \mathcal{T}_D} |K_{\tau} * F| \right\|_{L^p(U)}.$$

Proof. Let $k = 2$. For any given cube $\mathbf{q} \in \mathcal{Q}_{D^2}$, take $V'_1, \dots, V'_A \in \mathcal{V}_{k-1}$ to be the minimizer obeying

$$(2.8) \quad \sup_{\tau \in \mathcal{T}_D(V'_1, \dots, V'_A)} \int_{\mathbf{q}} |K_{\tau} * F|^p = \min_{V_1, \dots, V_A \in \mathcal{V}_k} \sup_{\tau \in \mathcal{T}_D(V_1, \dots, V_A)} \int_{\mathbf{q}} |K_{\tau} * F|^p.$$

Using partition of unity and Minkowski inequality, we have

$$(2.9) \quad \|F\|_{L^p(U)} \leq \left(\sum_{\mathbf{q}} \left\| \sum_{\tau \in \mathcal{T}_D(V'_1, \dots, V'_A)} K_{\tau} * F \right\|_{L^p(\mathbf{q})}^p \right)^{\frac{1}{p}} + \left(\sum_{\mathbf{q}} \left\| \sum_{\tau \notin \mathcal{T}_D(V'_1, \dots, V'_A)} K_{\tau} * F \right\|_{L^p(\mathbf{q})}^p \right)^{\frac{1}{p}}.$$

The first term is clearly bounded by

$$(2.10) \quad D^{2n} \|F\|_{\mathfrak{R}_{k,p,A,D}(U)}.$$

Note that there are only $O(A)$ many τ 's that are not in $\mathcal{T}_D(V'_1, \dots, V'_A)$. Henceforth, the second term in (2.9) is majorized by

$$(2.11) \quad CA \left\| \sup_{\tau \in \mathcal{T}_D} |K_\tau * F| \right\|_{L^p(U)}.$$

Therefore the proof is done. \square

Lemma 2.3. *Let K_1, \dots, K_n be dyadic numbers obeying that K_1 is the largest dyadic number not exceeding $R^{\varepsilon^{L/2}}$ and for every $j \in \{2, \dots, n\}$, K_j is the smallest dyadic number exceeding $K_{j-1}^{\varepsilon^{-\sqrt{L}/2}}$. For any integer $m \in \{2, \dots, n\}$, let A'_m denote $A_1^{\frac{1}{2^{m-1}}}/2$. For any Schwartz function F with Fourier support in Σ_R and any cube $U \in \mathcal{Q}_{R^2}$,*

$$(2.12) \quad \|F\|_{L^p(U)} \leq CA_1 \left\| \sup_{\tau \in \mathcal{T}_{K_1}} |K_\tau * F| \right\|_{L^p(U)} + K_m^{3n} \|F\|_{\mathfrak{R}_{m+1,p,A'_m,K_m}(U)} + \sum_{j=2}^m K_j^{\varepsilon^{10}} \Xi_{j,p}(A'_m, K_j, U).$$

Proof. Applying Lemma 2.2 with $D = K_1$ and $A = A_1$, we get

$$(2.13) \quad \|F\|_{L^p(U)} \leq K_1^{2n} \|F\|_{\mathfrak{R}_{2,p,A_1,K_1}(U)} + CA_1 \left\| \sup_{\tau \in \mathcal{T}_{K_1}} |K_\tau * F| \right\|_{L^p(U)}.$$

For any $j \in \{2, \dots, m\}$, use Lemma 2.1 with $k = j$, $D = K_j$, $A_j = A_1^{\frac{1}{2^{j-1}}}$ and $M = \sqrt{A_{j-1}}$ to obtain

$$(2.14) \quad \|F\|_{\mathfrak{R}_{j,p,A_{j-1},K_{j-1}}(U)} \leq K_j^{2n} \|F\|_{\mathfrak{R}_{j+1,p,A_j,K_j}(U)} + K_1 \Xi_{j,p}(\sqrt{A_{j-1}}/2, K_j, U).$$

Since L is a sufficiently large constant, (2.13) and (2.14) yield (2.12) as desired. \square

3. WAVE PACKET DECOMPOSITION

Let ω denote the smallest $\frac{1}{D} \times \dots \times \frac{1}{D} \times \frac{1}{D^2}$ rectangular box that contains 5θ for $\theta \in \Theta_D$. The collection of all such ω 's is written as Ω_D . For any measurable function f , define $P_\omega f$ by

$$(3.1) \quad \widehat{P_\omega f}(\xi) = \widehat{f}(\xi) \Phi_\omega(\xi) \text{ for } \xi \in \mathbb{R}^{n+1},$$

where Φ_ω is a bump function, supported on 2ω and taking value 1 in ω . Let F be a measurable function whose Fourier support is in Σ_D , so that $F = \sum_\theta F_\theta$. Without loss of generality, we can assume that θ 's in Θ_D are separated, that is, for any two distinct $\theta, \theta' \in \Theta_D$, the distance between τ_θ and $\tau_{\theta'}$ exceeds $100/D$. Otherwise, just simply partition F into $O(1)$ many subfunctions. Clearly, we have

$$(3.2) \quad F_\theta = P_\omega F.$$

Let T represent a dual box of 10ω . We take those dual boxes T 's for a given 10ω to tile \mathbb{R}^{n+1} . By partition of unity, there are Schwartz functions ϕ_T 's such that each ϕ_T is nonnegative and bounded by 1, Fourier transform $\widehat{\phi_T}$ is supported in a $\frac{1}{D} \times \dots \times \frac{1}{D} \times \frac{1}{D^2}$ rectangular box, centered at the origin, with the same direction as ω , and it enjoys

$$(3.3) \quad |\phi_T(\mathbf{x})| \leq C(1 + \text{dist}(\mathbf{x}, T))^{-100},$$

and

$$(3.4) \quad \sum_T \phi_T(\mathbf{x}) = 1,$$

for any $\mathbf{x} \in \mathbb{R}^{n+1}$. Here the sum (3.4) is taken over all pairwise disjoint T 's with the same direction. We denote this collection of T 's by S_ω . The function ϕ_T is similar to a bump function supported on T and it obeys

$$(3.5) \quad \left| \sum_{T: T \in S_\omega} \phi_T \right|^p \sim \sum_{T: T \in S_\omega} \phi_T^p,$$

due to the nonnegativity of ϕ_T and that ϕ_T 's are essentially disjointly supported with ω fixed. Using (3.2) and (3.4), we obtain

$$(3.6) \quad F(\mathbf{x}) = \sum_{\omega \in \Omega} \sum_{T \in S_\omega} \phi_T(\mathbf{x}) P_\omega F(\mathbf{x}).$$

We call $T \times \omega$ a tile and let \mathbf{S} denote the collection of all tiles $T \times \omega$'s with $\omega \in \Omega_D$ and $T \in S_\omega$. For $s = T \times \omega$, define

$$(3.7) \quad \Pi_s F(\mathbf{x}) = \phi_T(\mathbf{x}) P_\omega F(\mathbf{x}),$$

then we end up with a representation for F ,

$$(3.8) \quad F = \sum_{s \in \mathbf{S}} \Pi_s F.$$

For $s = T_s \times \omega_s$, $\Pi_s F$ is a function almost supported in T_s , with Fourier support in 2ω . Thus it can be viewed as a constant in T_s . By utilizing Bernstein's inequality, it is easy to get, for any $2 \leq p < \infty$,

$$(3.9) \quad \|\Pi_s F\|_p \sim |T_s|^{\frac{1}{p} - \frac{1}{2}} \|\Pi_s F\|_2 \sim D^{\frac{n+2}{p} - \frac{n+2}{2}} \|\Pi_s F\|_2.$$

For any dyadic number $\mu > 0$, $\mathbf{S}[\mu]$ represents a set containing every tile $s \in \mathbf{S}$ such that $\frac{\mu}{2} \leq \|\Pi_s F\|_2 \leq \mu$, and let F_μ denote

$$(3.10) \quad F_\mu = \sum_{s \in \mathbf{S}[\mu]} \Pi_s F.$$

For any measurable set U , we can localize F_μ in U by setting

$$(3.11) \quad F_{\mu, U} = \sum_{s \in \mathbf{S}_U[\mu]} \Pi_s F$$

where $\mathbf{S}_U[\mu] = \{s \in \mathbf{S}[\mu] : T_s^* \cap U \neq \emptyset\}$.

Lemma 3.1. *Then for any $B \in \mathcal{Q}_{\tilde{D}^2}$ with $\tilde{D} \geq D^{1+\varepsilon}$, the l^p -decoupling norm of $F_{\mu, B}$ satisfies*

$$(3.12) \quad \|F_{\mu, B}\|_2 \sim \mu |\mathbf{S}_B[\mu]|^{1/2}$$

$$(3.13) \quad \|F_{\mu, B}\|_{p, D, B^*} \sim \mu D^{\frac{n+2}{p} - \frac{n+2}{2}} |\mathbf{S}_B[\mu]|^{1/p}.$$

and

$$(3.14) \quad \sum_{\mu} \|F_{\mu, B}\|_{p, D, B^*} \leq CD^\varepsilon \|F\|_{p, D, B^*}.$$

Proof. (3.12) is a direct consequence of the orthogonality and (3.5). From (3.5) and (3.9), it follows that

$$(3.15) \quad \|F_{\mu,B}\|_{p,D,B^*}^p \sim \sum_{\omega} \sum_{\substack{s \in \mathbf{S}_B[\mu] \\ \omega_s = \omega}} \|\Pi_s F\|_p^p \sim \mu^p D^{n+2-(n+2)p/2} |\mathbf{S}_B[\mu]|,$$

which yields (3.13). We only need to focus those μ with $\frac{1}{D^{1/\varepsilon}}\|F\|_2 \leq \mu \leq 2\|F\|_2$, because $\|\Pi_s F\|_2 \leq \|F\|_2$ and all $\mu \leq 1/D^{1/\varepsilon}\|F\|_2$ terms contribute insignificant. Since there are only $O(\log D)$ many μ 's making significant contribution, to get (3.14), it suffices to prove that

$$(3.16) \quad \|F_{\mu}\|_{p,D,B} \leq C\|F\|_{p,D,B}.$$

Using (3.5), (3.9) and (3.15), we obtain

$$(3.17) \quad \|F\|_{p,D,B^*}^p \geq C \sum_{\omega} \sum_{s \in \mathbf{S}_{\omega}} \|\Pi_s F\|_{L^p(B^*)}^p \geq C \sum_{\omega} \sum_{\substack{s \in \mathbf{S}_{\omega} \\ T_s^* \cap B \neq \emptyset}} \|\Pi_s F\|_p^p \geq C \|F_{\mu,B}\|_{p,D,B^*}^p,$$

as desired. \square

For any $\omega \in \Omega_D$, $e(\omega)$ denotes the unit vector that is parallel to the longest side of its dual box. The (k,p) -broadness of F now can be adjusted in terms of its wavepacket decomposition (3.8). In fact, it can be defined by, for any cube B in \mathcal{Q}_{D^2} and $U \subset B$,

$$(3.18) \quad \|F\|_{\mathbf{Br}_{k,p}(A,U,B)} = \left(\sum_{\substack{\mathbf{q} \in \mathcal{Q}_{D_0^2} \\ \mathbf{q} \subset U}} \min_{V_1, \dots, V_A \in \mathcal{V}_{k-1}} \sup_{\tau \in \mathcal{T}_{D_0}(V_1, \dots, V_A)} \int_{\mathbf{q}} |K_{\tau} * \left(\sum_{s \in \mathbf{S}_{D,V}} \Pi_s F \right)(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}},$$

where

$$(3.19) \quad \mathbf{S}_{D,V} = \{s \in \mathbf{S} : \omega_s \in \Omega_D, \text{dist}(e(\omega_s), V) \leq 200D^{-1}\},$$

for some k -dimensional space $V \in \mathcal{V}_k$. V here may depend on F and B , but independent of τ , \mathbf{q} , U and the variable \mathbf{x} . In (3.18), we use the same (k,p) -broadness notation in the left side, since the right side is essentially the same as the (k,p) -broadness of F defined in (1.4). From (1.7) and (3.18), it is easy to get

$$(3.20) \quad \left\| \sum_{\mu} F_{\mu} \right\|_{\mathbf{Br}_{k,p}(A,U,B)} \leq CD^{\varepsilon} \sum_{\mu} \|F_{\mu}\|_{\mathbf{Br}_{k,p}(A/L,U,B)}.$$

This together with (3.14) indicates that we only need to pay attention to the function F associated to a given size μ , when decoupling.

4. L^2 ESTIMATES OF THE RESTRICTED (k,p) -BROADNESS

In this section, we build up some local estimates for the restricted broadness. We use p_k to denote $\frac{2k}{k-1}$.

Lemma 4.1. *Let $B \in \mathcal{Q}_{D^2}$ and $p \geq p_k$.*

$$(4.1) \quad \|F\|_{\mathbf{Br}_{k,p}(A,B)} \leq \frac{CD^{\varepsilon^2}}{D^{1+(n-k+1)(\frac{1}{2}-\frac{1}{p})}} \|F\|_2.$$

Lemma 4.1 follows from Guth's estimates for the (k,p) -broadness (see Proposition 8.1 in [5] with $m = k$). Using this result, we can obtain the following local L^2 -estimate.

Lemma 4.2. *Let $B \in \mathcal{Q}_{D^2}$, $Q \in \mathcal{Q}_D$ and $p \geq p_k$.*

$$(4.2) \quad \|F\|_{\mathbf{Br}_{k,p}(A,Q,B)} \leq \frac{CD^{\varepsilon^2}}{D^{\frac{1}{2}+(n-k+1)(\frac{1}{2}-\frac{1}{p})}} \|F\|_{L^2(Q^*)}.$$

Proof. Let V be the k -dimensional space in the definition (3.18) and V^\perp be its orthogonal complement in \mathbb{R}^{n+1} . We use Q_V to represent the smallest cube in the coordinate system (V, V^\perp) that contains Q . $\sqrt{D}\mathbb{Z}^{n-k+1}$ denotes the collection of lattice points that can be written as $(\sqrt{D}n_1, \dots, \sqrt{D}n_{n-k+1})$ in V^\perp for some integers n_1, \dots, n_{n-k+1} . Let \mathcal{V} stand for a family of k -dimensional planes that can be written as $V + \mathbf{x}$ for some lattice point $\mathbf{x} \in \sqrt{D}\mathbb{Z}^{n-k+1}$. Here $V + \mathbf{x}$ means a shift of V along V^\perp to the point \mathbf{x} . Using those planes in \mathcal{V} , the cube Q_V can be partitioned into a union of rectangular boxes J 's. This collection of J 's is denoted by $\mathcal{J}_{Q,V}$. Every $J \in \mathcal{J}_{Q,V}$ is a subset of Q_V , of dimensions about $\underbrace{\sqrt{D} \times \dots \times \sqrt{D}}_{n-k+1} \times \underbrace{D \times \dots \times D}_k$ in the system (V^\perp, V) . Thus there are at most $O(D^{\frac{n-k+1}{2}})$ such boxes in $\mathcal{J}_{Q,V}$. On each $J \in \mathcal{J}_{Q,V}$, (4.1) yields

$$(4.3) \quad \|F\|_{\mathbf{Br}_{k,p}(A,J,B)} \leq \frac{C}{D^{\frac{1}{2}+\frac{n-k+1}{2}(\frac{1}{2}-\frac{1}{p})}} \|F\|_{L^2(J^*)}.$$

for any $B \in \mathcal{Q}_{D^2}$ and $p \geq p_k$. Moreover, utilizing these J 's, we have

$$(4.4) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,Q,B)}^p \leq \sum_{J \in \mathcal{J}_{Q,V}} \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,J,B)}^p.$$

For any $Q \in \mathcal{Q}_D$, we set

$$(4.5) \quad F_{Q,V,\mu} = \sum_{s \in \mathbf{S}_{D,V} \cap \mathbf{S}_Q[\mu]} \Pi_s F.$$

$F_{Q,V,\mu}$ is a function with Fourier support in Σ_D , thus its Fourier transform is also supported in $\Sigma_{\sqrt{D}}$. Using the wave-packet decomposition (3.8) at scale \sqrt{D} , we write, on Q ,

$$(4.6) \quad F_{Q,V,\mu} = \sum_{\mu'} \sum_{s \in \mathbf{S}_{\sqrt{D},V} \cap \mathbf{S}_Q[\mu']} \Pi_s(F_{Q,V,\mu}) + \text{negligible term},$$

Here in the sum there are at most $O(\log D)$ many μ' . The reason we can localize the function in the cube Q is because the (k, p) -broadness $\|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,Q,B)}$ is restricted in Q . The constraint $s \in \mathbf{S}_{\sqrt{D},V}$ is due to Fourier support of $F_{Q,V,\mu}$. Because of this constraint, on each $J \in \mathcal{J}_{Q,V}$,

$$(4.7) \quad F_{Q,V,\mu} = \sum_{\mu'} \sum_{s \in \mathbf{S}_{\sqrt{D},V} \cap \mathbf{S}_J[\mu']} \Pi_s(F_{Q,V,\mu}) + \text{negligible term}.$$

Recall that we use $(F_{Q,V,\mu})_{\mu'}$ to denote $\sum_{s \in \mathbf{S}_{\sqrt{D},V} \cap \mathbf{S}_J[\mu']} \Pi_s(F_{Q,V,\mu})$. Apply the L^2 estimate (4.3) with $p \geq p_k$ to get

$$(4.8) \quad \|(F_{Q,V,\mu})_{\mu'}\|_{\mathbf{Br}_{k,p,\lambda}(\frac{A}{L}, J, Q)} \leq \frac{C}{D^{\frac{1}{2}} D^{\frac{n-k+1}{2}(\frac{1}{2}-\frac{1}{p})}} \left(\sum_{s \in \mathbf{S}_{\sqrt{D},V} \cap \mathbf{S}_J[\mu']} \|\Pi_s(F_{Q,V,\mu})\|_2^2 \right)^{1/2}.$$

On Q , $F_{Q,V,\mu}$ can be viewed as a constant along V^\perp -direction because of its Fourier support. Thus we have

$$(4.9) \quad \sum_{s \in \mathbf{S}_{\sqrt{D},V} \cap \mathbf{S}_J[\mu']} \|\Pi_s(F_{B,V,\mu})\|_2^2 \sim \frac{1}{D^{\frac{n-k+1}{2}}} \sum_{s \in \mathbf{S}_{\sqrt{D},V} \cap \mathbf{S}_Q[\mu']} \|\Pi_s(F_{B,V,\mu})\|_2^2.$$

From (4.8) and the property of equidistribution (4.9), we can sum up all $J \in \mathcal{J}_{Q,V}$ to get

$$(4.10) \quad \sum_{J \in \mathcal{J}_{Q,V}} \|(F_{Q,V,\mu})_{\mu'}\|_{\mathbf{Br}_{k,p,\lambda}(\frac{A}{L}, J, Q)}^p \leq \frac{CD^{\varepsilon^2}}{D^{\frac{p}{2}} D^{(n-k+1)(\frac{p}{2}-1)}} \left(\sum_{s \in \mathbf{S}_Q[\mu']} \|\Pi_s(F_{B,V,\mu})\|_2^2 \right)^{p/2},$$

which is bounded by, via the orthogonality in L^2 ,

$$(4.11) \quad \frac{CD^{\varepsilon^2}}{D^{\frac{p}{2}} D^{(n-k+1)(\frac{p}{2}-1)}} \left(\int_{Q^*} \sum_{s \in \mathbf{S}_Q[\mu]} \|\Pi_s F\|_2^2 \right)^{p/2} = \frac{CD^{\varepsilon^2}}{D^{\frac{p}{2}} D^{(n-k+1)(\frac{p}{2}-1)}} \|F\|_{L^2(Q^*)}^p,$$

as desired. \square

5. l^p -DECOUPLING AT μ -LEVEL

In this section, we provide a proof of Theorem 1.1. Theorem 1.1 can be reduced to the following proposition because of (3.14).

Proposition 5.1. *Let D be a sufficiently large dyadic number. Then for any $B \in \mathcal{Q}_{D^2}$ and $pk \geq p \geq \frac{2(n+1-\frac{k-1}{2})}{n-\frac{k-1}{2}}$,*

$$(5.1) \quad \|F_\mu\|_{\mathbf{Br}_{k,p}(A,B)} \leq CD^{n-\frac{2(n+1)}{p}+\varepsilon} \|F_{\mu,B}\|_{p,D,B^*}.$$

Here C is an absolute constant independent of μ , B , A , D and F .

We now further localize the broadness according to the size contribution from the function. Let $\lambda > 0$ be any dyadic number. For any $\frac{1}{D_0}$ -cap τ , $B \in \mathcal{Q}_{D^2}$ and a given k -dimensional subspace V determined by B , we define a level set as following,

$$(5.2) \quad E_{\lambda,\tau,B,V}[\mu] := \{\mathbf{x} \in \mathbb{R}^{n+1} : \frac{\lambda}{2} < |K_\tau * F_{B,V,\mu}(\mathbf{x})| \leq \lambda\}$$

where

$$(5.3) \quad F_{B,V,\mu} = \sum_{s \in \mathbf{S}_{D,V} \cap \mathbf{S}_B[\mu]} \Pi_s F.$$

On each $\mathbf{q} \in \mathcal{Q}_{D_0^2}$, set

$$(5.4) \quad \nu_{k,p,A,\lambda,\mu,F}(\mathbf{q}) := \min_{V_1, \dots, V_A \in \mathcal{V}_{k-1}} \sup_{\tau \in \mathcal{T}_{D_0}(V_1, \dots, V_A)} \int_{\mathbf{q} \cap E_{\lambda,\tau,B,V}[\mu]} |K_\tau * F_{B,V,\mu}(\mathbf{x})|^p d\mathbf{x}.$$

For any dyadic numebr λ , we define a local version of (k, p) -broadness by, for any $B \in \mathcal{Q}_{D^2}$,

$$(5.5) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,U,B)} := \left(\sum_{\mathbf{q} \in \mathcal{Q}_{D_0^2} : \mathbf{q} \subset U} \nu_{k,p,A,\lambda,\mu,F}(\mathbf{q}) \right)^{1/p}.$$

$\|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,B)}$ denotes $\|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,B,B)}$. There are only $O(\log D)$ many λ 's contributing significantly. We always assume that $\lambda \geq D^{-L} \|F_{\mu,B}\|_2$, since otherwise the problem is trivial. Henceforth, via a use of (1.7) for $O(\log D)$ many such λ 's, Proposition 5.1 can be reduced to

Proposition 5.2. *Let D be a sufficiently large dyadic number. Then for any $B \in \mathcal{Q}_{D^2}$ and $p_k \geq p \geq \frac{2(n+1-\frac{k-1}{2})}{n-\frac{k-1}{2}}$,*

$$(5.6) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,B)} \leq CD^{n-\frac{2(n+1)}{p}+\varepsilon} \|F_{\mu,B}\|_{p,D,B^*}.$$

Here C is an absolute constant independent of μ, λ, B, A, D and F .

For any dyadic number γ and any cube $B \in \mathcal{Q}_{D^2}$, a cube $Q \in \mathcal{Q}_D$ is called a γ -cube associated to $\mathbf{S}_B[\mu]$ if the number of $s \in \mathbf{S}_B[\mu]$ with $T_s^* \cap Q^* \neq \emptyset$ is between $\gamma/2$ and γ . For any $B \in \mathcal{Q}_{D^2}$, we define B_γ be union of all γ -cubes in \mathcal{Q}_D associated to $\mathbf{S}_B[\mu]$ that are contained in B .

Lemma 5.1. *The number of γ -cubes $Q \in \mathcal{Q}_D$ associated to $\mathbf{S}_B[\mu]$ is at most $\frac{CD^{1+\varepsilon}|\mathbf{S}_B[\mu]|}{\gamma}$.*

Proof. First we have that

$$(5.7) \quad \gamma \sum_{Q:\gamma\text{-cubes}} |Q| \leq \sum_Q \int_Q \sum_{s \in \mathbf{S}_B[\mu]} \mathbf{1}_{2T_s^*}(\mathbf{x}) d\mathbf{x},$$

because $2T_s^*$ contains Q whenever T_s^* touches Q^* . By Fubini, the right side of (5.7) can be bounded by

$$(5.8) \quad \sum_{s \in \mathbf{S}_B[\mu]} \sum_{Q: Q \cap 2T_s^* \neq \emptyset} |Q|.$$

Lemma 5.1 now follows from (5.7), (5.8) and the fact that there are at most $D^{1+\varepsilon}$ many Q intersecting a fixed tube T_s^* . \square

Because we only need to pay attention to at most $O(\log D)$ many γ 's. To get Proposition 5.2, it suffices to prove the following proposition for a fixed γ .

Proposition 5.3. *For any cube $B \in \mathcal{Q}_{D^2}$ and any $p_k \geq p \geq \frac{2(n+1-\frac{k-1}{2})}{n-\frac{k-1}{2}}$,*

$$(5.9) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,B_\gamma,B)} \leq CD^{n-\frac{2(n+1)}{p}+\varepsilon} \|F_{\mu,B}\|_{p,D,B^*}.$$

Here C is an absolute constant independent of $\mu, \lambda, \gamma, B, A, D$ and F .

We now start to prove this proposition.

Lemma 5.2. *Let $p > 2$. For any λ with condition*

$$(5.10) \quad \lambda \leq \mu D^{-\frac{n+2}{2}} D^{\frac{pn-2(n+1)}{p-2}}.$$

Then for any $B \in \mathcal{Q}_{D^2}$,

$$(5.11) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,B_\gamma,B)} \leq CD^{n-\frac{2(n+1)}{p}} \|F_{\mu,B}\|_{p,D,B^*}.$$

Proof. Notice that for any $\mathbf{q} \in \mathcal{Q}_{D^2_0}$,

$$(5.12) \quad \nu_{k,p,A,\lambda,\mu,F}(\mathbf{q}) \sim \lambda^{p-2} \nu_{k,2,A,\lambda,\mu,F}(\mathbf{q}),$$

which, by the definition of the local k -broadness in (5.5) and then a trivial L^2 -estimate, implies

$$(5.13) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,B_\gamma,B)}^p \sim \lambda^{p-2} \|F_\mu\|_{\mathbf{Br}_{k,2,\lambda}(A,B_\gamma,B)}^2 \leq \lambda^{p-2} \mu^2 |\mathbf{S}_B[\mu]|,$$

which is bounded by, from the condition (5.10) and (3.13),

$$(5.14) \quad D^{pn-2(n+1)} \mu^p D^{-(p-2)\frac{n+2}{2}} |\mathbf{S}_B[\mu]| \sim D^{pn-2(n+1)} \|F_{\mu,B}\|_{p,D,B^*}^p,$$

as desired. \square

Lemma 5.3. *Let $2 \leq p < p_k$. Suppose that λ obeys*

$$(5.15) \quad \lambda \geq \mu D^{-\frac{n+2}{2}} \left(\frac{\gamma^{\frac{1}{k-1}}}{D^{np-2(n+1)}} \right)^{\frac{1}{p_k-p}}$$

For any $B \in \mathcal{Q}_{D^2}$,

$$(5.16) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,B_\gamma,B)} \leq C D^{n-\frac{2(n+1)}{p}+\varepsilon} \|F_{\mu,B}\|_{p,D,B^*}.$$

Proof. Observe that for any $\mathbf{q} \in \mathcal{Q}_{D_0^2}$,

$$(5.17) \quad \nu_{k,p,A,\lambda,\mu,F}(B) \sim \lambda^{p-p_k} \nu_{k,p_k,A,\lambda,\mu,F}(B),$$

which, by the definition of the local k -broadness in (5.5), implies

$$(5.18) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,Q,B)}^p \sim \lambda^{p-p_k} \|F_\mu\|_{\mathbf{Br}_{k,p_k,\lambda}(A,Q,B)}^{p_k},$$

for any γ -cube Q . Using (4.2) with $p = p_k$, we then obtain

$$(5.19) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,Q,B)}^p \leq C \lambda^{p-p_k} \frac{\|F_{\mu,Q}\|_2^{p_k}}{D^{p_k+(n-k+1)\left(\frac{p_k}{2}-1\right)}} \leq C \lambda^{p-p_k} \frac{\mu^{p_k} \gamma^{\frac{p_k}{2}}}{D^{p_k+(n-k+1)\left(\frac{p_k}{2}-1\right)}},$$

which, by invoking the condition (5.15) of λ , is bounded by

$$(5.20) \quad D^{np-2(n+1)} \mu^p \gamma D^{n+1-\frac{n+2}{2}p}.$$

Then for any cube $B \in \mathcal{Q}_{D^2}$, via the definition of B_γ and Lemma 5.1, we get

$$(5.21) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,B_\gamma,B)}^p \leq C D^{np-2(n+1)+\varepsilon} \mu^p D^{n+2-\frac{n+2}{2}p} |S_B[\mu]| \sim C D^{np-2(n+1)+\varepsilon} \|F_{\mu,B}\|_{p,D,B^*}^p.$$

In the last step, (3.13) is utilized. \square

Lemma 5.4. *Suppose that $p_k > p \geq \frac{2(n+1-\frac{k-1}{2})}{n-\frac{k-1}{2}}$ and $\gamma \leq D^{k-1}$. Then λ satisfies either (5.10) or (5.15).*

Proof. It follows from the fact that

$$(5.22) \quad \left(\frac{\gamma^{\frac{1}{k-1}}}{D^{np-2(n+1)}} \right)^{\frac{1}{p_k-p}} \leq D^{\frac{pn-2(n+1)}{p-2}}$$

provided that $\gamma \leq D^{k-1}$ and $p \geq \frac{2(n+1-\frac{k-1}{2})}{n-\frac{k-1}{2}}$. \square

Now Proposition 5.3 is a consequence of Lemma 5.2, Lemma 5.3, and Lemma 5.4. Henceforth Proposition 5.1 follows.

6. AN APPLICATION OF THEOREM 1.1

$\mathbf{R}(f)$ represents the restriction operator given by

$$(6.1) \quad \mathbf{R}(f)(x, t) = \int_{\mathbb{R}^n} \phi(\xi) e^{ix \cdot \xi} e^{it|\xi|^2} f(\xi) d\xi,$$

for $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. Here ϕ is a standard bump function on a 1-neighborhood away from the origin.

In [5], Guth proved the following two theorems.

Theorem 6.1 (Guth). *Let $U \in \mathcal{Q}_{R^2}$.*

$$(6.2) \quad \|\mathbf{R}(f)\|_{L^p(U)} \leq R^\varepsilon \|f\|_{L^p}.$$

provided that

$$(6.3) \quad p > \frac{2(3n+4)}{3n} \text{ for } n \text{ even},$$

$$(6.4) \quad p > \frac{2(3n+5)}{3n+1} \text{ for } n \text{ odd}.$$

Theorem 6.2 (Guth). *Let $1 \leq m \leq n$ and $U \in \mathcal{Q}_{R^2}$.*

$$(6.5) \quad \|\mathbf{R}(f)\|_{\mathfrak{B}_{m+1,p,A'_m,K_m}(U)} \leq CR^\varepsilon \|f\|_2.$$

provided that $p \geq \frac{2(n+m+2)}{n+m}$.

In this section, we present that Theorem 1.1 together with Guth's k -broad restriction estimates yields Theorem 6.1. Let $\beta_p \geq \epsilon$ denote the best constant such that

$$(6.6) \quad \|\mathbf{R}(f)\|_{L^p(B)} \leq K^{\beta_p} \|f\|_p,$$

for any sufficiently large dyadic number K any $B \in \mathcal{Q}_{K^2}$. Let $U \in \mathcal{Q}_{R^2}$. Notice that on U , $\mathbf{R}(f)$ is a function with Fourier support in Σ_R . Applying (2.12), we control $\|\mathbf{R}(f)\|_{L^p(U)}$ by

$$(6.7) \quad CA_1 \left\| \sup_{\tau \in \mathcal{T}_{K_1}} |K_\tau * (\mathbf{R}(f))| \right\|_{L^p(U)} + K_m^{3n} \|\mathbf{R}(f)\|_{\mathfrak{B}_{m+1,p,A'_m,K_m}(U)} + \sum_{j=2}^m K_j^{\varepsilon^{10}} \Xi_{j,p}(f)(A'_m, K_j, U)$$

for any $m \in \{2, \dots, n\}$. Here K_j 's are defined as in Lemma 2.3, and $\Xi_{j,p}(f)(A'_m, K_j, U)$ is given by

$$(6.8) \quad \left(\sum_{B \in \mathcal{Q}_{K_j^2}: B \subset U} \|\mathbf{R}(f)\|_{\mathbf{B}_{j,p}(A'_m, B, B)}^p \right)^{\frac{1}{p}}.$$

Lemma 6.1. *Let $p \geq \frac{2(n+1)}{n}$. Then*

$$(6.9) \quad \left\| \sup_{\tau \in \mathcal{T}_{K_1}} |K_\tau * (\mathbf{R}(f))| \right\|_{L^p(U)} \leq \left(\frac{R}{\sqrt{K_1}} \right)^{\beta_p} \|f\|_p.$$

Proof. Let f_τ denote the restriction of f to the cap τ . Then $K_\tau * (\mathbf{R}(f))$ is essentially $\mathbf{R}(f_\tau)$. Using parabolic rescaling, it is easy to see that

$$(6.10) \quad \left\| \sup_{\tau} \mathbf{R}(f_\tau) \right\|_{L^p(U)} \leq \left(\sum_{\tau} \|\mathbf{R}(f_\tau)\|_{L^p(U)}^p \right)^{1/p} \leq K_1^{\frac{2(n+1)}{p}-n} \left(\frac{R}{\sqrt{K_1}} \right)^{\beta_p} \|f\|_p.$$

□

Lemma 6.2. *Suppose that*

$$(6.11) \quad \|\mathbf{R}(f)\|_{\mathbf{Br}_{j,p}(A'_m, B, B)} \leq CK_j^{n-\frac{2(n+1)}{p}+\varepsilon^2} \|\mathbf{R}(f)\|_{p, K_j, B^*}.$$

Then

$$(6.12) \quad \Xi_{j,p}(f)(A'_m, K_j, U) \leq CK_j^{\varepsilon^2} \left(\frac{R}{\sqrt{K_j}} \right)^{\beta_p} \|f\|_p.$$

Proof. (6.11) yields that

$$(6.13) \quad \Xi_{j,p}(f)(A'_m, K_j, U) \leq CK_j^{n-\frac{2(n+1)}{p}+\varepsilon^2} \|\mathbf{R}(f)\|_{p, K_j, U}.$$

(6.12) follows from the definition of the decoupling norm and the parabolic rescaling (6.10). □

Theorem 6.1 can be obtained by using Theorem 6.2 with $m = n/2$ if n is even and $m = \frac{n+1}{2}$ if n is odd, and utilizing Theorem 1.1 with $k = m$.

7. ESTIMATES ASSOCIATED TO DIFFERENT SCALES

In this section, we start with a function F with Fourier support in Σ_K and consider the l^p decoupling of the restricted (k, p) -broadness of F . Like what we did in the previous section, we only need to pay our attention to decoupling the localized broadness $\|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A, B_\gamma, B)}$ for $B \in \mathcal{Q}_{K^2}$. Here the localized broadness is given by (5.5) with $D = K$, and B_γ is the union of all γ -cubes in \mathcal{Q}_K associated to $\mathbf{S}_B[\mu]$ that are contained in the cube B . $F_{B,V,\mu}$ is defined in (5.3) with $D = K$. From (4.2), we have for $p \geq p_k$,

$$(7.1) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A, Q, B)} \leq \frac{C}{\sqrt{K} K^{(n-k+1)(\frac{1}{2}-\frac{1}{p})}} \|F_{B,V,\mu}\|_{L^2(Q^*)}.$$

$F_{B,V,\mu}$ is a function with Fourier support in Σ_K , thus its Fourier transform is also supported in Σ_D for any dyadic number $D \leq \sqrt{K}$. Using the wave-packet decomposition (3.8) at scale D , we write, on Q ,

$$(7.2) \quad F_{B,V,\mu} = \sum_{\mu'} \sum_{s \in \mathbf{S}_{D,V} \cap \mathbf{S}_Q[\mu']} \Pi_s(F_{B,V,\mu}) + \text{negligible term},$$

Here in the sum there are at most $O(\log D)$ many μ' . The reason we can localize the function in the cube Q is because the (k, p) -broadness $\|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A, Q, B)}$ is restricted in Q . The constraint $s \in \mathbf{S}_{D,V}$ is due to Fourier support of $F_{B,V,\mu}$.

Lemma 7.1. *Let Q be a γ -cube in $\mathbf{S}_B[\mu]$, and let $\mathbf{S}'_Q[\mu']$ denote $\mathbf{S}_{D,V} \cap \mathbf{S}_Q[\mu']$. For any μ' ,*

$$(7.3) \quad (\mu')^2 |\mathbf{S}'_Q[\mu']| \leq K^{\varepsilon^2} K^{-1} \mu^2 \gamma.$$

Proof. From the orthogonality in $L^2(Q^*)$, we have

$$(7.4) \quad \|F_{B,V,\mu}\|_{L^2(Q^*)}^2 \sim \sum_{\mu'} \sum_{s \in \mathbf{S}'_Q[\mu']} \|\Pi_s(F_{B,V,\mu})\|_2^2 \leq K^{\varepsilon^2} K^{-1} \sum_{s \in \mathbf{S}_Q[\mu]} \|\Pi_s F\|_2^2 \sim K^{\varepsilon^2} K^{-1} \mu^2 \gamma.$$

since Q is a γ -cube. (7.3) now follows from the fact that

$$(7.5) \quad \sum_{s \in \mathbf{S}'_Q[\mu']} \|\Pi_s(F_{B,V,\mu})\|_2^2 \sim (\mu')^2 |\mathbf{S}'_Q[\mu']|.$$

□

Lemma 7.2. *For any γ -cube $Q \in \mathcal{Q}_K$ in $\mathbf{S}_B[\mu]$, $B \in \mathcal{Q}_{K^2}$, any dyadic number $D \leq \sqrt{K}$ and $p_k \geq p \geq 2$, we have*

$$(7.6) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,Q,B)}^p \leq C \left(\frac{\mu \gamma^{\frac{1}{2}}}{\lambda K^{\frac{n+2}{2}}} \right)^{p_k - p} D^{(k-1)(\frac{p}{2}-1)+\varepsilon} \|F_{B,V,\mu}\|_{p,D,Q^*}^p.$$

Proof. Apply (7.1) for $p = p_k$ to get

$$(7.7) \quad \|F_\mu\|_{\mathbf{Br}_{k,p_k,\lambda}(A,Q,B)} \leq \frac{C}{\sqrt{K} K^{(n-k+1)(\frac{1}{2}-\frac{1}{p_k})}} \|F_{B,V,\mu}\|_{L^2(Q^*)}$$

For any fixed μ , we have

$$(7.8) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,Q,B)}^p \sim \lambda^{p-p_k} \|F_\mu\|_{\mathbf{Br}_{k,p_k,\lambda}(A,Q,B)}^{p_k},$$

which is majorized by, from the p_k estimate (7.7) and then the wave-packet decomposition (7.2),

$$(7.9) \quad \frac{C \lambda^{p-p_k}}{K^{\frac{p_k}{2}} K^{(n-k+1)(\frac{p_k}{2}-1)}} \left(\sum_{\mu'} (\mu')^2 |\mathbf{S}'_Q[\mu']| \right)^{p_k/2} + \text{negligible term}.$$

Now using (3.13) and (3.14), we have

$$(7.10) \quad (\mu')^{p_k} |\mathbf{S}'_Q[\mu']|^{\frac{p_k}{2}} \leq (\mu')^{p_k-p} |\mathbf{S}'_Q[\mu']|^{\frac{p_k}{2}-1} D^{(n+2)(\frac{p}{2}-1)+\varepsilon} \|F_{B,V,\mu}\|_{p,D,Q^*}^p.$$

Let μ^* be defined by

$$(7.11) \quad (\mu^*)^2 |\mathbf{S}'_Q[\mu^*]| = \sup_{\mu'} (\mu')^2 |\mathbf{S}'_Q[\mu']|.$$

Here recall that there are only $O(\log D)$ many μ' 's to which we need to pay attention. Henceforth, from (7.9) and (7.10) with $\mu' = \mu^*$, we end up with

$$(7.12) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,Q,B)}^p \leq \frac{C \lambda^{p-p_k} D^{(n+2)(\frac{p}{2}-1)+\varepsilon}}{K^{\frac{p_k}{2}} K^{(n-k+1)(\frac{p_k}{2}-1)}} (\mu^*)^{p_k-p} |\mathbf{S}'_Q[\mu^*]|^{\frac{p_k}{2}-1} \|F_{B,V,\mu}\|_{p,D,Q^*}^p,$$

which is bounded by, by (7.3),

$$(7.13) \quad \frac{C \lambda^{p-p_k} D^{(n+2)(\frac{p}{2}-1)+\varepsilon}}{K^{\frac{p_k}{2}} K^{(n-k+1)(\frac{p_k}{2}-1)}} \mu^{p_k-p} \gamma^{\frac{p_k-p}{2}} \frac{1}{K^{(p_k-p)/2}} |\mathbf{S}'_Q[\mu^*]|^{\frac{p}{2}-1} \|F_{B,V,\mu}\|_{p,D,Q^*}^p,$$

Because there are at most $K^{n+1}/D^{n+2-\varepsilon}$ many $D \times \dots \times D \times D^2$ tubes in Q with the same direction and the number of directions does not exceed D^{k-1} , we have

$$(7.14) \quad |\mathbf{S}'_Q[\mu^*]| \leq D^{k-1} \frac{K^{n+1}}{D^{n+2-\varepsilon}}.$$

Now (7.6) simply follows from (7.13) and (7.14). \square

Proposition 7.1. *For any $B \in \mathcal{Q}_{K^2}$, any dyadic number $D \leq \sqrt{K}$ and $p_k \geq p \geq 2$, we have*

$$(7.15) \quad \|F_\mu\|_{\mathbf{Br}_{k,p,\lambda}(A,B_\gamma,B)} \leq C \left(\frac{\mu\gamma^{\frac{1}{2}}}{\lambda K^{\frac{n+2}{2}}} \right)^{\frac{p_k-p}{p}} D^{(k-1)(\frac{1}{2}-\frac{1}{p})+\sqrt{\varepsilon}} \|F_{B,V,\mu}\|_{p,D,B^*}.$$

Proof. This is an immediate consequence of Lemma 7.2 by summing up all γ -cubes Q 's. \square

An l^p -decoupling of F with Fourier support in Σ_K was proved by Bourgain and Demeter in [2] for $p > \frac{2(n+2)}{n}$. Any improvement of the lower bound of p plus a use of Proposition 7.1 brings a new estimate in the restriction conjecture. This is the reason why we keep Proposition 7.1 in the paper.

REFERENCES

- [1] J. Bourgain and C. Demeter, *The proof of the l^2 Decoupling Conjecture*, Annals of Math. 182 (2015), no. 1, 351-389.
- [2] J. Bourgain and C. Demeter, *l^p decouplings for hypersurfaces with nonzero Gaussian curvature*, ArXiv: 1407.0291.
- [3] J. Bourgain and L. Guth, *Bounds on Oscillatory Integral Operators Based on Multilinear Estimates* G.A.F.A., Vol. 21, No. 6 (2011), 1239-1295.
- [4] L. Guth, *A restriction estimate using polynomial partitioning*, J. Amer. Math. Soc. 29 (2016), no. 2, 371C-413.
- [5] L. Guth, *Restriction estimates using polynomial partitioning II*, ArXiv: 1606.07682
- [6] I. Laba, T. Wolff, *A local smoothing estimate in higher dimensions*, J. Anal. Math. 88 (2002) 149-171, dedicated to the memory of Tom Wolff.

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